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THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. III. MAY, 1896. No. 5.

A SPECIAL COMPLEX OF THE SECOND DEGREE AND ITS RELATION WITH THE PENCILS OF CIRCLES.

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1. Before entering upon the treatment of this problem I will make a few preliminary remarks which, although well known to the reader, may give a clearer conception of what follows.

If a straight line is given we can write the equations of its projections upon the co-ordinate planes in the following form :*

$$(1)\qquad \begin{aligned} L&=yZ-zY, \\ M&=zX-xZ, \\ N&=xY-yX, \end{aligned}$$

where x, y, z designate the current co-ordinates. The six constants L, M, N, X, Y, Z can be considered as the co-ordinates of the straight line and satisfy the relation

$$(2)\qquad LX+MY+NZ=0.$$

An algebraic complex of straight lines of the n^{th} degree is defined by an equation of the form

$$(3)\qquad F(L,\, M,\, N,\, X,\, Y,\, Z)=0,$$

F being a polynom of the n^{th} degree and homogeneous in L, M, N, X, Y, Z.

Through every point in spaces passes an infinite number of lines belonging to the complex ; they form a cone of the n^{th} order, and in every plane lies an infinite number of lines belonging to the complex and enveloping a curve of the

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*We use the designation of M. Picard in his *Traite d'Analyse*, Vol. I, p. 312.

n^{th} class. Thus, a special kind of a complex of the n^{th} degree may be obtained by all the tangents of a surface of the n^{th} order, or the secants of a curve in space of the n^{th} order.

2. In our problem we define as a special complex P of the second degree the system of secants passing through a fixed conic. Every point in space determines a cone of the second order, whose elements belong to the complex and every plane intersects the conic in two points which represent a degenerated curve of the second class, whose tangents belong to the complex.

The conic itself we will describe in the following manner:

Through any two fixed points A and B of the xy-plane draw the two circles

$$(4)\qquad U_1=(x-a_1)^2+(y-b_1)^2-r_1{}^2=0,$$

$$(5)\qquad U_2=(x-a_2)^2+(y-b_2)^2-r_2{}^2=0,$$

and form the pencil of circles

$$(6)\qquad U_1-\lambda U_2=0$$

passing through A and B. At the center of every circle of the pencil erect a perpendicular to the xy-plane and equal to the radius of the circle above and below the xy-plane. The extremities of these perpendiculars lie in an equilateral hyperbola H whose plane passes through the central line of the pencil of circles and is perpendicular to tne xy-plane. The vertices of the hyperbola are equal distant from the xy-plane and lie in a perpendicular through the center of the circle with the sect AB as a diameter.*

To every point of the equilateral hyperbola belongs a circle of the pencil (6), which with the point determines a right cone whose elements all include angles of 45° with the xy-plane. We may ask what is the character of the system of lines R passing through the equilateral hyperbola and including angles of 45° with the xy-plane. For this purpose intersect the *cone-director* of these lines with the plane at infinity and establish the new complex Q consisting of all the lines passing through the intersection. As the intersection is a circle I, the complex is of the second degree and contains all the lines including angles of 45° with the plane xy.

Evidently the system R is the common solution of the complex P and Q and is therefore a *congruence*. The degree of this congruence is 6, since the degrees of P and Q are 2, and since the hyperbola H and the circle I have two points in common. Through each point in space pass two lines, and in each plane lie four lines belonging to the congruence. It is therefore of the second order and of the fourth class. Since the equilateral hyperbola H is symmetrical in regard to the xy-plane, it is easily seen that the complex and the congruence connected with it are symmetrical to the xy-plane, in other words they are reflected into themselves. It is known that through every generatrix of a congruence of straight lines pass two developable surfaces whose elements belong to

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*The thought to represent points in space by circles in a plane originated with Prof. W. Fiedler, of Zurich, who applied it in his beautiful treatise on "*Cyclographie*," Teubner, Leipzig.

the congruence. In our congruence R the developable surfaces through a generatrix D are the right cone having its vertex in the hyperbola H and the hyperbolic cylinder passing through H. The focal surface of the congruence degenerates into the hyperbola H and the plane at infinity. If we designate the representation of a pencil of circles by Fiedler's method, and the complex and congruence of rays connected with it as *cyclographic*, we may now state the theorem:

The theory of the pencils of circles is identical with the theory of the cyclographic congruence.

3. To the first pencil of circles through A and B, or cyclographic congruence, we add another pencil of circles through the points C and D, or cyclographic congruence. It may be determined by two circles

$$V_1=(x-c_1)^2+(y-d_1)^2-s_1{}^2=0, \tag{7}$$

$$V_2=(x-c_2)^2+(y-d_2)^2-s_2{}^2=0, \tag{8}$$

passing through C and D and assumes the form

$$V_1-\mu V_2=0. \tag{9}$$

The corresponding congruence is obtained as in the first pencil. Designating this congruence by S and the complex through the hyperbola G which represents the pencil (9) by T we have to solve the problem to find the common part of the congruences R and S, or as these have the circle I at infinity in common, to find the common figure of the complexes P, Q, and T. Each of the hyperbolas H and G intersect the circle I in two points and as the complexes are all of the second degree, they have a ruled surface in common whose degree according to the rules of algebra is

$$2\times2\times2\times2-2\times2-2\times2=8.$$

To a generatrix in this ruled surface of the eighth order can be found one in the same surface symmetrical to the first in regard to the xy-plane. Hence the whole surface is symmetrical to the xy-plane and as it contains two double generatrices through the circular points of the circle I, it intersects the xy-plane in a bicircular curve of the fourth order. Every generatrix of the surface intersects the xy-plane in a point of the curve and includes an angle of 45° with the xy-plane.

Through each generatrix pass four developable surfaces, two hyperbolic cylinders and two cones of the second order. These cones are tangent to each other and intersect the xy-plane in two tangent circles. As these circles always pass through A, B and C, D and as their point of tangency lies in the above curve we have the theorem:

The locus of the points of tangency of each two tangent circles of two pencils of circles is a bicircular curve of the fourth order.

Figure 1 will show the relation of these pencils in the case that each two circles are tangent.

4. We will now take another view of the problem. For fixed values of λ and μ the equations of two circles respectively belonging to the pencil (6) and (9) may be written

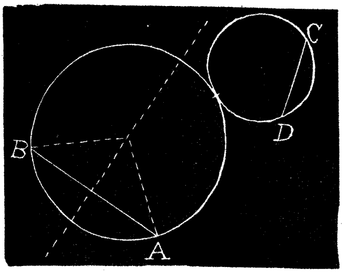

Fig. 1.

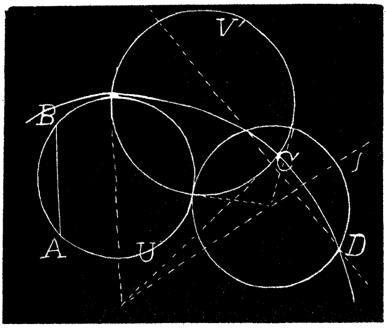

Fig. 2.

$$(10)\qquad x^2+y^2-2\frac{a_1-\lambda a_2}{1-\lambda}x-2\frac{b_1-\lambda b_2}{1-\lambda}y+\frac{M_1-\lambda M_2}{1-\lambda}=0,$$

$$(11)\qquad x^2+y^2-2\frac{c_1-\mu c_2}{1-\mu}\quad-2\frac{d_1-\mu d_2}{1-\lambda}y+\frac{N_1-\mu N_2}{1-\mu}=0,$$

where $\qquad M_1=a_1{}^2+b_1{}^2-r_1{}^2,\ M_2=a_2{}^2+b_2{}^2-r_2{}^2,$

$$N_1=c_1{}^2+d_1{}^2-s_1{}^2,\ N_2=c_2{}^2+d_2{}^2-s_2{}^2.$$

The condition that the circle (10) is orthogonal to the circle (11) is

$$2\,\frac{a_1-\lambda a_2}{1-\lambda}\cdot\frac{c_1-\mu c_2}{1-\mu}+2\,\frac{b_1-\lambda b_2}{1-\lambda}\cdot\frac{d_1-\mu d_2}{1-\mu}-\frac{M_1-\lambda M_2}{1-\lambda}-\frac{N_1-\mu N_2}{1-\mu}=0.$$

It is now possible to determine the co-efficients of this equation such that for variable parameters the pencils (10) and (11) are projective. In this case each two corresponding circles are orthogonal.

Evidently we have to put $\mu=\lambda$, which after some reductions gives for the equation of condition

$$[2a_1c_1+2b_1d_1-M_1-N_1]-\lambda[2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2]$$
$$+\lambda^2[2a_2c_2+2b_2d_2-M_2-N_2]=0. \qquad (12)$$

This indicates that in the first place the circles (4) and (7), and (5) and (8) must be orthogonal. Secondly, for every value of λ there must be

$$2a_2c_1+2b_2d_1-M_2-N_1+2a_1c_2+2b_1d_2-M_1-N_2=0.$$

This equation is satisfied if the circles (5) and (7), and (4) and (8) are orthogonal. In this case the pencils (6) and (9) are said to be conjugate pencils of circles. Every circle of the one pencil is orthogonal to every other circle of the other. But in equation (12) we do not desire to change the points A, B, C, D. Thus, the equation can be satisfied by changing the radii of the circles (4), (5), (7), (8), which gives for the solutions of M_1, M_2, N_1, N_2 three equations with four unknown quantities. We can however fix one of the circles without altering the final result. This implies the theorem :

Two pencils of circles can be made projective in one and only one way such that corresponding circles in the projectivity are orthogonal.

The product of these projective pencils is a bicircular curve of the fourth order, as it is well known. In figure 2, we consider the two pencils of circles through A and B, and C' and D', where C' and D' are assumed to be imaginary and on the line l and in these pencils two orthogonal circles U and V' intersecting each other in two points J and J'. In these points draw tangent circles to U having their centers on l. These circles are orthogonal to V' and intersect each other in two fixed points C and D, *i. e.*, they belong to the conjugate pencil of circles of the pencil through C' and D'. Whence the general theorem :

The locus of the points of tangency of each two tangent-circles of two pencils of circles is a bicircular curve of the fourth order. The same curve is also produced by one of the pencils and the projective conjugate pencil of the other pencil.

Under the given conditions the equation of the curve may be written $U_1V_2'-U_2V_1'=0$.

It is easily seen that this curve passes through the four points A, B, C, D and as stated in the theorem contains the circular points at infinity as double-points.

5. Without entering into further details on the nature of this curve it may be mentioned that there exists an interesting connection between this curve and the circular curves of the third order if these are considered as locii of points from which two sects AB and CD appear under the same angle. An analogon exists in space, the discussion of which however goes over the limits of this article. A paper on this subject by the author was read in the January session of the Kansas Academy of Science and will appear in the next volume of the transactions of this Academy.

PROBLEMS.

1. Given n straight lines in a plane. Another straight line in this plane revolves about a fixed point and in every position intersects the n lines